Field line resonant frequencies and ionospheric conductance: Results from a 2-D MHD model

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[1] The magnetised plasma of the near-Earth space environment supports ultra-low frequency (ULF; 1–100 mHz), magnetohydrodynamic (MHD) oscillations. For sufficiently large ionospheric conductances, field line resonances (FLRs) form between the northern and southern ionospheres. These conditions are usually met for daytime ionosphere conductance values. The FLRs are normal modes of the system and may be used to remote sense plasma mass densities in the magnetosphere. The oscillations lose energy in the ionosphere whose properties determine the boundary conditions, particularly resonance damping effects. Using a two dimensional (2D) MHD model of the magnetosphere and realistic ionospheric boundary conditions, the variation in resonant frequency with ionosphere conductivity is reported. For typical mid to low latitude magnetosphere and realistic ionosphere boundary conditions, the FLRs change by less than 5%. This translates to an uncertainty of 7% in plasma mass density.


1. Introduction

[2] The Earth, in space, is immersed in the outer solar atmosphere. The geomagnetic field and solar wind interact to form the terrestrial magnetosphere where ultra low frequency (ULF; 1–100 mHz) plasma waves have wavelengths comparable with the magnetosphere dimensions. With suitable ionospheric boundary conditions the shear Alfvén mode forms field line resonances (FLRs) [Dungey, 1954; Tamao, 1964]. The variation of the geomagnetic field and cold plasma population throughout the magnetosphere gives a spatially dependent Alfvén speed. For most of the magnetosphere (L ≥ 1.6), when tracking out into space from the ionosphere in a direction along the geomagnetic field, the Alfvén speed is a minimum around the equatorial plane. This property allows for the estimation of the equatorial plasma mass density in the magnetosphere using FLRs [Obayashi and Jacobs, 1958; Gul'elmi, 1967].

[3] In order to obtain plasma mass density information, the FLR frequency at a given location must be known. Present ground based methods combine ULF wave data at ground locations with the solution to the MHD standing wave problem to provide equatorial plasma mass density estimates [e.g., Singer et al., 1981; Waters et al., 1996]. These standing wave solutions assume perfect reflection at the ionosphere boundaries. However, more realistic ionosphere boundaries introduce wave damping and other effects that may alter the FLR properties, including the frequency [Budnik et al., 1998; Yoshikawa and Itonaga, 1999]. In this paper, we describe a two dimensional (2D) numerical model of ULF wave dynamics in the magnetosphere that features realistic ionosphere boundary conditions, including field line curvature and the ionosphere Hall current. Using this model, we have investigated the effects of ionosphere boundary conditions on ULF resonant frequencies and hence, plasma mass density estimates.

[4] The fast and shear Alfvén magnetohydrodynamic (MHD) wave modes oscillate in the cold, magnetized magnetosphere plasma [Alfvén and Fälthammar, 1963]. The ionosphere is known to alter the amplitude and polarization properties of ULF waves [Nishida, 1964; Hughes, 1974; Hughes and Southwood, 1976]. One of the first effects to be studied in detail was the 90° rotation (NDR) of the ULF wavefields as they pass from the magnetosphere to the atmosphere [Hughes, 1974]. For ULF waves, the displacement current is small and the field aligned current, J, associated with the shear mode in the magnetosphere is related to the wave magnetic field, B, by \( \nabla \times \mathbf{B} = \mu_0 J \). However, in the neutral atmosphere, \( \nabla \times \mathbf{B} = 0 \) and a rotation of the wavefield into the direction of the horizontal component of the propagation vector, \( \hat{k} \) occurs [Hughes, 1983]. More recently, Sciffer et al. [2005] have shown that the NDR occurs for low and midlatitudes as well. This neatly explains why FLRs detected by spacecraft show magnetic field oscillations predominantly in the azimuthal components (east–west) [e.g., Takahashi and McPherron, 1984], while data from the north–south aligned sensors of ground magnetometers show FLR signatures.

[5] The formation of FLRs depends on the reflection coefficients associated with ULF waves and the ionosphere. In the electrostatic limit (\( \nabla \times \mathbf{E} = 0 \) in the ionosphere), the
reflection coefficient for the shear Alfvén mode is \[ [\text{Scholer}, 1970] \]

\[
A_{\text{static}} = \frac{\Sigma_a - \Sigma_p}{\Sigma_a + \Sigma_p} \tag{1}
\]

where the Alfvén wave conductance is \( \Sigma_a = \frac{1}{n e T_a} \) and \( \Sigma_p \) is the height integrated Pedersen conductivity. This is the simplest approximation for the wave reflection process and assumes a vertical geomagnetic field, horizontally uniform conductance and ignores any contribution from ionosphere Hall currents. While a number of studies have developed analytic expressions for reflection and transmission coefficients \([e.g., \text{Sorokin, 1970; Alperovich and Fedorov, 1992; Yoshikawa et al., 1996}], all treat the geomagnetic field as either vertical or horizontal \([\text{Zhang and Cole, 1994}].\)

[6] Progress toward a realistic ionosphere boundary formulation was presented by \text{Yoshikawa et al. [1996]} who discussed more complete expressions for the reflection and transmission coefficients, including the inductive Hall current feedback mechanism in magnetosphere-ionosphere coupling. Analytic solutions for the reflection and transmission coefficients in the presence of an oblique geomagnetic field were developed by \text{Sciffer and Waters [2002]} who approximated the ionosphere as a thin current sheet, allowing the application of standard electromagnetic boundary conditions. More detailed numerical modeling that used data from the International Reference Ionosphere (IRI) showed that a thin current sheet ionosphere approximation retained the important features of ULF wave passage from the magnetosphere to the atmosphere \([\text{Sciffer et al., 2005}].\)

[7] There have been some studies that have examined ULF wave behavior in two and three dimensional MHD models. A 2D, eigenvalue analysis by \text{Yoshikawa and Itonaga [1999]} showed how the height integrated Pedersen and Hall conductivities (or conductances) controlled the FLR eigenmodes. Their results highlighted the importance of the inductive ionosphere terms, a central idea when including the ionosphere Hall current. However, the model used by \text{Yoshikawa and Itonaga [1999]} contained a rectangular box geometry with a vertical, uniform geomagnetic field everywhere and constant Alfvén speed along the field. These approximations ignore the more realistic weighting of the Alfvén speed in the equatorial plane of the magnetosphere, over emphasizing effects on FLR properties near the ionosphere.

[8] Numerical studies of ULF wave properties in a 3D MHD formulation have been described by \text{Lee and Lysak [1989, 1991]}. This model features a dipole geomagnetic field. A study of FLR damping using this 3D formulation, compared with GOES-6 satellite data was described by \text{Budnik et al. [1998]}. The simulation allowed for field line curvature and included realistic Alfvén speed profiles. However, the inner boundary was set at 5 \( R_E \) with the ionosphere located at 2 \( R_E \). Furthermore, the ionosphere reflection was described by equation (1) which ignores the ionosphere Hall current and oblique geomagnetic field. In order to understand effects of the ionosphere-atmosphere system on ULF wave properties in the magnetosphere, we need to have a realistic description of the ionosphere boundary conditions. In the present paper we incorporate the more complete physics of ULF wave reflection and transmission described by \text{Sciffer and Waters [2002]} into a 2D numerical MHD model.

2. MHD Model Description

[9] ULF waves incident from the magnetosphere into the ionosphere and underlying atmosphere may be described as an electromagnetic disturbance. We assume that the magnetosphere and ionosphere plasmas are electrically neutral and that the zero order electric field, \( \vec{E}_0 \), is zero. The relevant Maxwell equations are

\[
\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E} \tag{2}
\]

\[
\epsilon_\perp \frac{\partial \vec{E}_\perp}{\partial t} = \frac{1}{\mu_0} \left( \vec{\nabla} \times \vec{B} \right)_\perp \tag{3}
\]

where \( \vec{B} \) and \( \vec{E} \) are the wavefields, \( \epsilon_\perp = \epsilon_0 \left( 1 + \frac{\omega^2}{\omega_{pe}^2} \right) \), the dielectric constant for the plasma. Assuming an ideal MHD plasma in the magnetosphere, the field aligned component of the electric field is zero \((i.e., \vec{E}_\parallel = 0).\)

[10] The geomagnetic field of the inner magnetosphere \((\leq 5 \ R_E)\) may be described by a dipole geometry. The orthogonal dipole coordinate system has a field aligned unit vector, an azimuthal unit vector that is orthogonal to the field direction and the cross product completes the set. The wavelengths of the oscillations are much larger than the thickness of the ionosphere so height integrated conductivities are used and the ionosphere is represented by a thin current sheet at an altitude, \( R_f = 120 \) km. Differences in the horizontal components of the perturbation magnetic fields just above (magnetosphere) and just below (atmosphere) the current sheet must be consistent with the horizontal ionosphere current. Therefore spherical coordinates are more appropriate for describing the situation at the ionosphere. We need a coordinate system that is spherical for the ionosphere becoming dipolar as we move out into space. One solution is to use two sets of basis vectors where one set is tangential to the geomagnetic field while the other set is orthogonal to the ionospheric current sheet. \text{Proehl et al. [2002]}, \text{Lysak [2004]} and \text{Rankin et al. [2006]} contain descriptions of this nonorthogonal basis. We now describe our adaptation to this formulation.

[11] We define a coordinate set that is a function of the spherical coordinate system. Consider three coordinate functions, \( u^1(r, \theta, \phi), u^2(r, \theta, \phi) \) and \( u^3(r, \theta, \phi) \). We are interested in two basis vector sets that can be computed from these functions. If \( \vec{R} \) is the position vector in spherical coordinates then the tangential basis vectors are defined by

\[
e_i = \frac{\partial \vec{R}}{\partial u^i} = \partial_i \vec{R} \tag{4}
\]

while the cotangent (reciprocal, dual or gradient) basis vectors are

\[
e^i = \nabla u^i \tag{5}
\]

[12] In an orthonormal basis system these two coordinate systems coincide. In general, each basis vector points in a
extend it into a full meridional slice of the magnetosphere.

In this paper we use the coordinate system of \( R_E \) space, \((r, \theta, \phi)\), for a 2D description. All distances are scaled in units of \( R_E \) (the Earth’s radius), \( \theta \) is the colatitude while \( \theta_0 \) is the colatitude at the Earth surface in the northern hemisphere after tracing along the magnetic field from \( p(r, \theta) \). Therefore \( \theta_0 \) is given by \( \cos \theta_0 = \sqrt{1 - R_f/L} = \sqrt{1 + \frac{u^2}{R}} \). In this paper we use the coordinate system of Lysak [2004] and extend it into a full meridional slice of the magnetosphere.

Equations (2) and (3) may be reformulated using the covariant and contravariant basis vectors [e.g., \( D'haeseleer \) et al., 1991]. In this system, the curl of a vector \( A \) is given by

\[
\nabla \times A = \frac{1}{J} \epsilon^{ijk} e_i \partial_j A_k
\]

where \( \epsilon^{ijk} \) is the antisymmetric permutation tensor (0 if any two indices are the same and \( \pm 1 \) for cyclic/anticyclic permutation of the indices) and \( J = e_1 \cdot (e_2 \times e_3) \) is the Jacobian for the covariant system. We assume an azimuthal dependence of \( \epsilon^{(m)ij} \) for all perturbation fields where \( m \) is the azimuthal wave number \([\text{Olson and Rostoker, 1978}]. \) In this system, \( e_3 \) is tangential to the geomagnetic field so that \( E_3 = 0 \) for an ideal MHD approximation. The corresponding component equations are

\[
\frac{\partial E_1}{\partial t} = \frac{V^2}{J} (\partial_3 B_2 - imB_3)
\]

\[
\frac{\partial E_2}{\partial t} = \frac{V^2}{J} (\partial_3 B_1 - \partial_1 B_3)
\]

\[
\frac{\partial B_1}{\partial t} = \frac{1}{J} (\partial_3 E_2 - \partial_2 E_1)
\]

\[
\frac{\partial B_2}{\partial t} = \frac{1}{J} (imE_1 - \partial_1 E_2)
\]

\[
\frac{\partial B_3}{\partial t} = \frac{1}{J} (\partial_3 E_1 - \partial_1 E_2)
\]

where \( V^2 = \frac{1}{\rho e_3} \).

To calculate the covariant from the contravariant components we use the metric tensors. The mappings are

\[
E_1 = g_{11} E^1, \quad E_2 = g_{22} E^2
\]

\[
B_1 = g_{11} B^1 + g_{13} B^3, \quad B_2 = g_{22} B^2
\]

\[
B_3 = g_{31} B^1 + g_{33} B^3
\]

The system of equations (11) to (15) were solved over a staggered grid so that all differences were centered. The time dependence in the variables was solved by a leapfrog technique where the electric fields were solved at time, \( t_n \). The covariant components of the electric field were then evolved and rotated into the contravariant representation using (16). The magnetic fields were evaluated at time \( t_{n+\frac{1}{2}} \) and the covariant terms were calculated from (17) for the start of the next time step.

2.1. Boundary Conditions for the Magnetosphere

The geometry and solution grid is shown in Figure 1. On the inner L shell boundary (\( L = 1.2 \)), \( E_3 \) and \( B_1 \) were set to zero. Using (12), this also sets \( \partial_1 B_3 = 0 \). This boundary
perfectly reflects the oscillations. The outer boundary condition is

\[ B_3 = \Gamma(u_3, t) \text{ for } 0 \leq t \leq t_d \]  
\[ = 0 \text{ for } t > t_d \]  

where \( \Gamma \) is a function of time and distance along the outer field line of the model. The time, \( t_d \) specifies the duration that the outer field line was driven. For \( t < t_d \) we impose a field aligned perturbation along the outer field line, producing a compressional component into the solution at the outer boundary. After \( t > t_d \) the boundary condition sets the compression component of the magnetic field on the outer field line to zero.

2.2. Boundary Conditions at the Ionosphere

[18] Boundaries at the northern and southern ionospheres correspond to values of \( u^3 = \pm 1 \) where the discontinuity in the wave magnetic field is consistent with the ionospheric currents. The ionospheric current density is described by \( \hat{\Sigma} \cdot \vec{E} \). The current sheet is thin so that the radial current density is zero. The horizontal ionospheric electric fields, \( E_1 \) and \( E_2 \), and the radial magnetic field component, \( B_3 \) are continuous across the sheet. The ionospheric boundary is given by [e.g., Hughes, 1974; Yoshikawa and Itonaga, 2000; Sciffer and Waters, 2002; Sciffer et al., 2004]

\[ \mu_0 \hat{\Sigma} \cdot \vec{E} = \vec{r} \times \Delta \vec{B} \]  

where \( \hat{\Sigma} \) is the height integrated conductivity tensor and \( \Delta \vec{B} \) is the discontinuity in the magnetic field across the current sheet. For an oblique geomagnetic field, the conductivity tensor is

\[ \Sigma = \begin{bmatrix} \Sigma_0 \Sigma_{\rho} & \Sigma_0 \Sigma_{H} \cos \alpha \\ \Sigma_{\rho} \Sigma_{H} \cos \alpha & \Sigma_{\rho} \Sigma_{Z} \cos \alpha \\ \Sigma_0 \Sigma_{H} \sin \alpha & \Sigma_0 \Sigma_{Z} \sin \alpha \\
\Sigma_{Z} \Sigma_{Z} & \Sigma_0 \Sigma_{H} \sin \alpha \\
\Sigma_0 \Sigma_{Z} \cos \alpha & \Sigma_0 \Sigma_{Z} \sin \alpha \end{bmatrix} \]  

where \( \Sigma_0 \), \( \Sigma_{\rho} \), and \( \Sigma_{H} \) are the height integrated Pedersen, Hall and direct conductivities respectively. The direct conductivity relates the geomagnetic field aligned electric fields and currents. The angle between the geomagnetic field and the radial direction, \( \alpha \) is given by

\[ \cos \alpha = -2 \cos \theta / \sqrt{1 + 3 \cos^2 \theta} \]  

while

\[ \Sigma_{Z} = \Sigma_0 \cos^2 \theta + \Sigma_\rho \sin^2 \theta. \]  

[19] The boundary condition (20) was implemented by specifying the spherical components of the perturbation electric and magnetic fields at the ionosphere. In the staggered grid, the covariant electric field components \( E_1 \) and \( E_2 \) and the contravariant \( B^2 \) component are evaluated along both the northern and southern ionosphere, while the horizontal magnetic field components \( (B_1 \text{ and } B_2) \) are evaluated one half-cell above the ionosphere. If the solution below the current sheet in the neutral atmosphere is known then the change in the magnetic field \( \Delta \vec{B} \) can be computed and (20) may be inverted to find the electric field.

2.3. Solution in the Atmosphere

[20] In order to invert (20) a solution in the atmosphere must be found. Assuming that the atmosphere is described by \( \nabla \cdot \vec{B} = \nabla \times \vec{B} = 0 \) then the magnetic field may be expressed in terms of a scalar potential, \( \Psi \) where \( \vec{B} = -\nabla \Psi \) and \( \nabla^2 \Psi = 0 \). We solve the Laplace equation in the atmosphere using spherical harmonic functions to obtain the magnetic field below the current sheet. For ULF waves, the ground is a good conductor so the radial component of the wave magnetic field at the ground is set to zero (i.e., \( B_r = \frac{\partial \Psi}{\partial r} = 0 \) at \( r = R_E \)). At the ionosphere current sheet, continuity in the radial magnetic field sets \( B_r = \frac{\partial \Psi}{\partial r} = h_r B^3 \) where \( h_r \) is the radial scale factor on the spherical harmonics at the ionosphere. Further details of the solution in the neutral atmosphere are described in the Appendix.

3. Results

[21] The properties of ULF wave propagation and resonance in the magnetosphere are largely determined by the spatial variation of the Alfvén speed. Estimates of the cold plasma composition and population as a function of radial distance in the equatorial plane may be determined from in situ measurements [e.g., Chappell et al., 1970] and given the dipole geomagnetic field approximation, the Alfvén speed in the equatorial plane is obtained. Another possibility is to use estimates of the FLRs as a function of latitude. Using measured values for the fundamental FLRs from low [e.g., Waters et al., 1991; Menk et al., 2000] through to high latitudes [Waters et al., 2006] and realistic magnetic field models, typical values for the Alfvén speed may be obtained. The Alfvén speed profile used in our model was constructed by a combination of these two approaches and is shown in Figure 2. The profile includes a plasmaopause as seen by the decrease in \( V_A \) around \( L = 4.5 \). Given the equatorial values for the plasma mass density, the off-equator variation along the geomagnetic field was given a radial dependence according to \( r^{-3} \). On the basis of satellite observations of Chappell et al. [1970], this appears to be a reasonable functional form.

[22] A simulation that excites broadband ULF energy in the modeled magnetosphere may be devised using the time and spatial dependence of the excitation source described by Lee and Lysak [1991]. The time dependence of the amplitude of the compressional excitation signal at the equatorial plane on the outer field line in our model is shown in Figure 3. The spectrum of this signal is similar to a low-pass filter response with 3dB attenuation at 100 mHz. The spatial structure in the east-west direction was specified by using an azimuthal wave number, \( m = 2 \). The subsequent time series are available at every grid location in the model. We focus on the signal that appears at grid points closest to the ground and those near the ionosphere.

[23] The model uses non-orthogonal coordinates as described in section 2 to allow for the oblique geomagnetic field at mid and low latitudes. We are interested in changing the ionosphere boundary and documenting any associated changes in the FLR frequencies. A first approximation that includes the ionosphere Hall current is to set a uniform
The ULF energy passes through the ionosphere and the way the perturbations interact with the plasmapause. The ionosphere compared with those threading the plasmasphere field lines have a greater proportion involved with the plasmasphere rather than the plasmatrough because plasma mass density estimates as the FLR shifts in latitude with ionosphere conductance. Figure 5 shows that the shift in latitude of the resonance amplitude peak is only a few degrees as the Pedersen conductance changes from 0.1 S to 10 S. In practice, it is difficult to isolate the amplitude peak in ground magnetometer records and the cross phase and/or amplitude difference spectra may be used. [Baransky et al., 1985; Waters et al., 1991].

In order to compute the cross phase and amplitude difference spectra, the time series from the grid points on both sides of 49° latitude were analyzed. This corresponds to a station separation of two degrees in latitude (∼220 km). The results for the different values for the Pedersen and Hall

![Figure 3](image-url)  
Figure 3. Time dependence of the field aligned magnetic perturbation applied to the outer field line of the model as the excitation signal.
Conductivities are shown in Figure 6. The cross phase spectra consistently show a peak around $\gamma_0 = 19\,\text{mHz}$, giving an error of $0.1\,\text{mHz}$, within experimental error for typical cross phase spectra. Similar spectra are obtained for a station separation of 1 degree ($\approx 110\,\text{km}$) but with the cross phase peak reduced.

The amplitude difference spectra show a zero crossing at slightly lower frequencies, $18\,\text{mHz}$ for the high conductance case reducing to $17\,\text{mHz}$ for the lower conductance values. At this latitude, the variation in Alfvén speed used in the model gives a change in resonant frequency of $2\,\text{mHz}$ per degree of latitude. If we experimentally estimated a $19\,\text{mHz}$ resonance peak in the cross phase spectrum to be located at $49^\circ$ latitude and solved the ULF wave equation assuming perfect ionosphere reflection then we would be in error by $1\,\text{mHz}$. At these latitudes, this translates to an error in equatorial plasma mass density of $7\%$.

In contrast to the resonance amplitude maximum, the peak in cross phase remains quite constant at $19\,\text{mHz}$ as the conductance changes. If we compare the process to a forced, damped simple harmonic system then we would expect the natural frequency to be greater than the resonant frequency, i.e., the frequency for maximum amplitude in a forced, damped system. The amount of damping can be estimated from the resonance width from Figure 4 for the various values of the conductance. Since the excitation pulse has a frequency response essentially flat around $20\,\text{mHz}$, we express the damping in terms of the half-width, the latitude range (in degrees) for the half power width.

In practice, the complication for estimating the amount of damping in a resonance from ground magnetometer data comes from the ‘spreading’ of the ULF spatial structure by spatial integration of the ionosphere currents. The results of this comparison are shown in Table 1. The resonance widths appearing just above the ionosphere differ from those obtained from the ground data for the more spatially localized resonances (larger values for the conductivity). A further factor that influences the spatial amplitude variation of FLRs detected on the ground is the Hall conductance. A smaller value for $\Sigma_H$ gives a smaller amplitude peak for an FLR with an associated smaller half

**Figure 4.** Power at 20 mHz in the east-west magnetic field perturbation versus latitude for grid points located just above the ionosphere for different values of ionosphere Pedersen conductance (solid:10 S, dotted:2 S, dash:1 S, dash dot:0.4 S, dash dot dot:0.1 S).

**Figure 5.** Power at 20 mHz in the north-south magnetic field perturbation versus latitude for grid points located just above the ionosphere for different values of ionosphere Pedersen conductance (solid:10 S, dotted:2 S, dash:1 S, dash dot:0.4 S).

**Figure 6.** (Top) Cross phase and (Bottom) amplitude difference spectra for the 20 mHz fundamental FLR for different values of ionosphere Pedersen conductance (dotted:10 S, dashed:2 S, dash dot:1 S, dash dot dot:0.4 S).
power level and wider latitude range at this level. For a given Pedersen conductance, a smaller Hall conductance does not reduce the amplitude of the FLR peak at grid points just above the ionosphere.

There are also approximations when estimating the plasma mass density from the observed FLRs that involve the solution for the wave equation. For illustration, assume that we have a large conductivity so that the resonance has a narrow half width in latitude and large amplitude (ionosphere damping effects are small). From Figure 5 we have detected a 20 mHz resonance at 48.8° latitude. Estimation of the equatorial plasma mass density now involves the solution to the ULF wave equation. The assumptions used in solving the wave equation involve special conditions for the plasma perturbations and the background magnetic field, $B$, and were discussed by Singer et al. [1981], Waters et al. [2006] and Kabin et al. [2006]. The solution assumes purely transverse oscillations with the azimuthal wave number set to zero to describe the toroidal, shear Alfven mode. There is no wave energy coupling to nearby field lines and no background electric currents (no $J \times B$ force). Singer et al. [1981] argued that any $J \times B$ effects on the wave equation are small for the Olson-Pfitzer field model.

A further inconsistency between the real magnetosphere and the wave equation formulation concerns the existence of uncoupled ULF wave modes in distorted field geometries. Wright and Evans [1991] have discussed the magnetic field topologies that can support uncoupled, cold plasma wave modes. Even in a dipole geometry, while purely toroidal oscillations decouple, the purely poloidal oscillations do not. For the 20 mHz resonance located at 48.8° latitude in a dipole field, the solution to the wave equation gives an equatorial plasma mass density of $7.46 \times 10^{-18}$ kg m$^{-3}$. The actual value used in the MHD model was $7.92 \times 10^{-18}$ kg m$^{-3}$. Therefore field distortions and coupled wave mode effects have only a small influence on plasma mass density estimates.

So far, we have only considered ULF oscillations that occur for the same conductivity values in both the northern and southern ionospheres. There have been studies that show how the resonance spatial structure along the field changes according to asymmetry in the ionosphere conductivity [e.g., Yagova et al., 1999]. Does the estimation for plasma mass density depend on whether the ionosphere conductivity differs in each hemisphere (season)? In order to investigate this, the IRI2001 model was used to generate the ionosphere conductivities versus latitude for 26 January 2000, typical solar maximum conditions with high summer in the southern hemisphere. The height integrated conductivities versus latitude for this case are shown in Figure 7.

In practice, different ionosphere conductivities in each hemisphere for summer/winter events would not preserve the symmetric plasma mass distribution along the geomagnetic field about the equatorial plane that we have in the model. While this is simple to change in the model, we are interested in whether the different hemisphere ionosphere conductivities significantly alter plasma mass density estimates. Keeping the same plasma mass distribution in the model magnetosphere provides a comparison that only includes changes in the ionosphere boundary conditions. The amplitude of the 20 mHz signal with latitude for this case is shown in Figure 8. The largest peak in amplitude occurs at 48.3° latitude compared with 48.8° for identical north and south conductivities.

5. Conclusions

We have developed a 2D MHD model of ULF wave propagation in the magnetosphere that includes features associated with a realistic description of the inner boundary. Using a non-orthogonal coordinate system, the dipolar geomagnetic field is meshed with the spherical ionosphere geometry to allow the application of standard electromagnetic boundary conditions. The oblique nature of the geomagnetic field at mid and low latitudes is included in

![Table 1. Resonance Widths (in Degrees Latitude) for the 20 mHz Frequency at 49° Latitude as a Function of Ionosphere Conductance](image)

![Figure 7. The height integrated Pedersen (solid), Hall (dotted) and direct (dashed; multiplied by $1 \times 10^{-7}$) conductivities versus latitude for typical southern hemisphere summer and solar maximum conditions.](image)
addition to inductive processes arising from ionosphere Hall currents.

[37] Our MHD model describes ULF wave propagation all the way to the Earth’s surface. This allows us to obtain modeled ground magnetometer signals, cross phase features for monitoring FLRs, and investigate how estimates of the resonance width may be different when measured in the ionosphere (e.g., using Doppler sounding) compared with ground magnetometer data. Table 1 shows that resonance widths estimated from ground data become more accurate as the conductivity decreases, spreading the spatial structure of the resonance. For typical daytime conductivities, resonance width estimates may be inflated to over twice the actual value.

[38] The improved MHD model was used to investigate how FLR frequencies and plasma mass density estimates change with ionosphere conductivity, including the Hall term. The Pedersen conductivity was changed from 0.1 to 10 S. This 100 fold increase in conductivity shifted a typical plasmasphere resonance by less than 2 degrees latitude. This is typical of experimental uncertainty when identifying the FLR frequency from cross phase spectra and translates to a 7% error in plasma mass density. Furthermore, while the amplitude peak shifted slightly with changing ionosphere conductivity, the cross phase peak remained relatively constant.

[39] The experimental procedure for estimating the plasma mass density relies on solving the ULF wave equation, given various simplifying assumptions. The dipolar geomagnetic field configuration coupled with the inclusion of the ionosphere Hall current in our model allows a comparison of how much these assumptions affect the plasma mass density estimates. Slightly lower (by 5%) values are obtained when solving the wave equation assuming perfect wave reflection at the ionosphere and a sole resonant ‘field line’.

Appendix A

A1. Basis Vectors

[40] The contravariant basis defined by $e^i = \nabla u^i$ are vectors which are normal to the plane $u^i = \text{constant}$ and are referred to as the normal basis. The contravariant basis vectors for the coordinate system given in (7)–(9) are

$$e^1 = \frac{R_I}{r} \sin \theta \left( \sin \theta \hat{\varphi} - 2 \cos \theta \hat{\varphi} \right)$$

$$e^2 = \frac{1}{r \sin \theta} \hat{\varphi}$$

$$e^3 = -\frac{R_I^2}{r^3 \cos^3 \theta} \left( \frac{\cos \theta}{2} \left(1 + 3 \cos^2 \theta \right) \hat{r} + \sin \theta \left(1 - \frac{R_I}{r} \right) \hat{\varphi} \right)$$

where $\hat{r}$, $\hat{\varphi}$, and $\hat{\theta}$ represent the unit vectors in spherical coordinates.

[41] The contravariant basis is defined by $e_i = \partial_i r$, where $\partial_i$ represents $\partial / \partial r$. These vectors are tangential to the $u^i$ and are referred to as the tangential basis vectors. The tangential basis vectors are

$$e_1 = \frac{r^2}{R_I \cos^2 \theta_0 (1 + 3 \cos^2 \theta)} \left( \left(1 - \frac{R_I}{r} \right) \hat{r} - \frac{\cot \theta (1 + 3 \cos^2 \theta_0) \hat{\phi}}{2} \right)$$

$$e_2 = r \sin \theta \hat{\phi}$$

$$e_3 = -\frac{r^3}{R_I^2 (1 + 3 \cos^2 \theta)} \left(2 \cos \theta \hat{\varphi} + \sin \theta \hat{\theta} \right)$$

[42] The $e^1$ and $e^2$ vectors are normal to the background magnetic field while $e_3$ is parallel to the background magnetic field. The $e_1$ and $e_2$ vectors are parallel to the ionospheric current sheet at $r = R_I$. The $u^3$ vector is directed radially outwards at the southern ionosphere and inward at the northern ionosphere.

A2. Neutral Atmosphere

[43] In the neutral atmosphere we seek solutions to the Laplace equation, $\nabla^2 \Psi = 0$, in spherical coordinates over a limited range of colatitudes. In general the solutions involve the separation of the radial, latitudinal and azimuthal dependences. For the latitudinal dependence, the boundary conditions which impose $\partial \Psi / \partial \varphi = 0$ at $\theta_l$ and $\theta_u$ (the lower and upper L shell boundary in each ionosphere) determine the appropriate Legendre polynomials. These are

$$P_l(\theta) = C_l P_{l0}(\cos \theta)$$

where $P_{l0}$ is a Legendre polynomial that has $v_l$ as a non integer. $v_l$ is the eigenvalue corresponding to the $l$’th eigen function, $P_l$ which satisfies the boundary conditions at $\theta_l$ and $\theta_u$. $C_l$ is a scale factor which is evaluated numerically such that the $P_l$ form an orthonormal basis set. The orthonormalization condition is

$$\int_{\theta_l}^{\theta_u} P_l(\theta) P_{l'}(\theta) d\theta = 1.$$
potential in the neutral atmosphere is

\[ \Psi(r, \theta) = \sum_l \left( A_l r^l + B_l r^{-(l+1)} \right) P_l(\theta) \]  

(A9)

[45] Using the radial boundaries the coefficients \( A_l \) and \( B_l \) are

\[ B_l = \frac{\nu_l}{\nu_l + 1} R_f^{2l+1} A_l, \]  

(A10)

[46] The \( A_l \) are evaluated using continuity of the radial magnetic field \( \frac{\partial B}{\partial r} = h_0 B^2 \) at the ionosphere by numerically solving for \( A_l \). That is

\[ A_l = \frac{1}{\nu_l R_f^{l-1}} \left( 1 - \frac{\nu_l}{\nu_l + 1} \right) \int_0^{\theta_0} B_l (R_f, \theta) P_l' (\theta) d\theta \]  

(A11)

[47] Once these have been determined, the magnetic field in the neutral atmosphere just below the ionosphere currents may be determined from the gradient of (A9) at \( r = R_f \). Equation (20) is then inverted to obtain the ionospheric electric fields. We may also evaluate the magnetic field at the ground from (A9) by evaluating the gradient of \( \Psi \) at \( r = R_E \).

A3. Scale Factors

[48] The scale factors at the ionospheric current sheet are used to rescale the fields calculated on the magnetospheric side of the ionosphere from their covariant and contravariant representation into a standard spherical coordinate basis. These scale factors are

\[ h_r = \frac{2 R_f \cos^2 \theta_0}{1 + 3 \cos^2 \theta_0} \]  

(A12)

\[ h_\theta = -\frac{R_f}{2 \sin \theta_0 \cos \theta_0} \]  

(A13)

\[ h_\phi = \frac{1}{R_f \sin \theta_0} \]  

(A14)

where \( \theta_0 \) is the invariant colatitude at the ionosphere. The scale factors are used in the following conversions at the ionosphere;

\[ B_r = h_r B^l \quad B_\theta = B_1/h_\theta \quad B_\phi = B_2/h_\phi \]  

(A15)

\[ E_r = E_1/h_\theta \quad E_\phi = E_2/h_\phi \]  

(A16)

[49] These allow the solution in the neutral atmosphere, which are solved in a spherical geometry, to be integrated into their non orthogonal representations in the magnetosphere.

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